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Extreme Point Linear Fractional Functional Programming

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Abstract: This paper deals with the optimization of the ratio of two linear functions subject to a set of linear constraints with the additional restriction that the optimal solution is to be an extreme point of another convex polyhedron. In this paper, an enumerative procedure for solving such type of problems is developed. For an illustration, a numerical example is also provided.

Introduction

Very recently a lot of work has been done in extreme point linear programming [*Kirby, Love* and *Swarup*, 1972, and 1970] wherein a linear objective function has been optimized over a convex polyhedron with the additional requirement that the optimal value should exist on an extreme point of another convex polyhedron. An extreme point technique has also been used in solving the Fixed Charge Problem and the Assignment Problem [*Puri* and *Swarup*, 1973/74, and *Puri*, w. y.]. Here, the objective function to be optimized is the ratio of two linear functions. This objective function which is neither convex nor concave, is optimized subject to linear inequalities with the additional constraint that the optimal solution should also be an extreme point of another convex polyhedron.

Some practical problems [*Swarup*, 1965, and *Martos*, 1964] of this type are zero-one integer fractional programming problem in which the requirement that each of the variables be either zero or one can be replaced by the condition that the optimal solution be an extreme point of the unit cube $IX \leq 1, X \geq 0$. Extreme point fractional functional programming problem is a larger class of problems than the class of zero-one integer fractional programming problems.

The most general mathematical form of extreme point linear fractional functional programming problem is:

$$\left.\begin{array}{l} \text{Max}\, Z = \dfrac{CX + \alpha}{DX + \beta} \\ \text{subject to } AX = b \\ \text{and } X \text{ is an extreme point of} \\ \qquad RX = t \\ \qquad\; X = 0 \end{array}\right| \qquad \ldots \text{Problem (I)}$$

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where C, D are $1 \times n$, X is $n \times 1$, A is $m \times n$, R is $p \times n$, b is $m \times 1$, t is $p \times 1$, 0 is $n \times 1$ and α, β are scaler constants.

The problem (I) is always bounded as any solution is to be an extreme point of $RX = t$, $X \geq 0$. We shall develop an enumerative technique in which extreme points of the convex polyhedron $AX = b$, $RX = t$, $X \geq 0$ are enumerated till we reach the required optimal solution. As the number of these extreme points is finite, the process will converge in a finite number of steps. When (I) has no solution, an indication to this effect will appear in the algorithm itself. The algorithm developed will be a dual method type of algorithm in the sense that feasibility and optimality of (I) are reached simultaneously.

The remaining sections of this paper contain the theory of the method for solving (I) and an algorithm based on this theory.

Theoretical Development:

To solve (I) we formulate another problem:

$$\left.\begin{array}{l} \text{Max } Z = \dfrac{CX + \alpha}{DX + \beta} \\ \text{subject to } FX = f \\ \qquad\qquad X \geq 0 \end{array}\right| \quad \ldots \text{Problem (II)}$$

$$\text{where } F = \begin{bmatrix} A \\ R \end{bmatrix} \text{ and } f = \begin{bmatrix} b \\ t \end{bmatrix}.$$

Problem (II) may be bounded or unbounded. In case it is unbounded, it can be made bounded by introducing a constraint of the form $1X \leq M$, M being sufficiently large positive number and 1 being sum vector, without losing any of its extreme points. So (II) will always be assumed to be bounded.

We shall also assume that

(i) the solution set $S = [X/FX = f, X \geq 0]$ of (II) is regular.

(ii) $DX + \beta > 0$ for all $X \in S$.

Under these assumptions optimal solution of (II) will occur at an extreme point of $FX = f$, $X \geq 0$ (i.e., optimal solution of (II) will be a basic feasible solution) [*Swarup*, 1965].

The extreme points of $FX = f$, $X \geq 0$ will be enumerated till we get a required optimal solution of (I) which is to be an extreme points of $RX = t$, $X \geq 0$.

Notations:

$J = [r_j/r_j \neq 0$, where r_j is j^{th} column of $R]$.

$J(X) = [r_j \in J/x_j \neq 0$ where $X = [x_1, x_2, \ldots, x_n]$ is a basic feasible solution of (II)].

$|J(X)|$ = number of elements in $J(X)$.

$S_1 = [X/AX = b$ and X is an extreme point of $RX = t$, $X \geq 0]$.

$S_2 = [X/X$ is an extreme point of $FX = f$, $X \geq 0]$.

$S = [X/FX = f, X \geq 0]$.

X_i = set of i^{th} best extreme point solutions of (II)

$= [X_{i1}, X_{i2}, \ldots, X_{is_i}]$.

$u_i = \dfrac{CX_{ij} + \alpha}{DX_{ij} + \beta}$ = value of the objective function at the elements of X_i.

B_i = Set of all the bases corresponding to the elements of X_i.

E_i = Set of all the bases adjacent to each element of B_i and yielding value of the objective function less than u_i.

(Two bases are said to be adjacent to each other if either of them is obtained from the other by changing only one of its vectors [*Hadley*, w. y., *Kirby*, *Love* and *Swarup*, 1970, and *Dantzig*, w. y.]).

Notice that $E_i \subseteq \bigcup_{j \geq i+1} (B_j)$.

H_i = Set of the bases adjacent to $I^{\text{st}}, II^{\text{nd}}, \ldots, i^{\text{th}}$ best extreme points of (II) leaving $I^{\text{st}}, II^{\text{nd}}, \ldots, i^{\text{th}}$ best extreme points of (II).

$= \bigcup_{j=1}^{i} E_j \Big\backslash \bigcup_{j=2}^{i} B_j$.

Note that $H_i \subseteq \bigcup_{j=1}^{i} E_j$.

Clearly $H_1 = E_1$ and $\bigcup_{j \geq l} B_j \subseteq H_{l-1}$).

ϕ = Null vector.

Note:

(i) The set X_i of i^{th} best extreme points of (II) is adjacent to some element of $\bigcup_{j=1}^{i-1} X_j$ [*Kirby*, *Love* and *Swarup*, 1972].

(ii) $X_i \in S_2 \Big\backslash \bigcup_{j=1}^{i-1} X_j$ and is such that

$$u_i \geq \frac{CX + \alpha}{DX + \beta}$$

where X is any extreme point belonging to $S_2 \Big\backslash \bigcup_{j=1}^{i-1} X_j$.

(iii) It is quite obvious that every point of S_1 is a point of S, converse may not be true. That is $S_1 \subseteq S$.

Theorem 1:

$S_1 \subseteq S_2$. That is, every extreme point of $RX = t$, $X \geq 0$ which is feasible for $AX = b$ is also an extreme point of $FX = f$, $X \geq 0$ [*Kirby*, *Love* and *Swarup*, 1972].

This theorem shows that the optimal solution of (I), if it exists, is given by some extreme point of (II). As S is regular, there will exist either of the following two possibilities.

(L. 1): (II) has a solution and (I) also has a solution and
(L. 2): (II) has a solution but (I) has no solution.

Firstly the case (L. 1) will be studied when (I) also has a solution and later the case (L. 2) is taken up when (I) has no solution.

As mentioned earlier, to solve (I) extreme points of (II) are to be enumerated in a certain order and the process is to be terminated when we get the required extreme point of $RX = t$, $X \geq 0$. That is, we will move systematically from one point to another point of S_2, till we reach the required extreme point which will be in S_1. So at each stage we have to see whether a point of S_2, obtained at that stage, is a point of S_1 or not. It is shown in *Kirby*, *Love* and *Swarup* [1972] that a point X of S_2 will be a point of S_1 if the number of non-null columns of R corresponding to non-zero basic variables in X is less than or equal to p and these columns are linearly independent provided ranks of F and R are respectively $m + p$ and p.

Procedure:

First solve (II) to find X_1 and u_1 [*Swarup*, 1965]. $X_1 \neq \phi$. If $X_1 \cap S_1 \neq \phi$, then any $X \in X_1 \cap S_1$ will be the required optimal solution of (I). But if $X_1 \cap S_1 = \phi$ (i.e., no element of X_1 is an extreme point of $RX = t$, $X \geq 0$), then an optimal solution of (I) must be an element of $S_1 \cap (S_2 \backslash X_1)$ since by theorem 1 an optimal solution of (I) must be an element of $S_1 \cap S_2$. Then proceed to find the set X_2 for which find B_1 and E_1. Bases of E_1 which yield the greatest value (say u_2) of the objective function, generate the set X_2. There will be a unique second best extreme point solution of (II) (i.e., X_2 will have a single element) if there is only one element of E_1 which yields u_2. If $X_2 = \phi$, (I) has no solution. If $X_2 \neq \phi$ and $X_2 \cap S_1 \neq \phi$, then any $X \in X_2 \cap S_1$ will be an optimal solution of (I). But if $X_2 \cap S_1 = \phi$, proceed to find X_3. To determine X_3, first find B_2, E_2 and H_2. Bases of H_2 which yield the greatest value (say u_3) of the objective function, generate X_3. If $X_3 = \phi$, (I) has no solution. If $X_3 \neq \phi$ and $X_3 \cap S_1 \neq \phi$, then any $X \in X_3 \cap S_1$ will be an optimal solution of (I). And if $X_3 \neq \phi$ and $X_3 \cap S_1 = \phi$, the process is repeated by finding $X_4, X_5, \ldots$ till for some i we get $X_i \neq \phi$ and $X_i \cap S_1 \neq \phi$ in which case any $X \in X_i \cap S_1$ will be an optimal solution of (I).

Observe that for each X_i obtained, $X_i \subseteq S_2$ and $X_i \neq \phi$. The procedure will converge in a finite number of steps since

(i) $S_1 \subseteq S_2$.
(ii) S_2 has a finite number of elements.
(iii) no X_i is repeated because $u_i > u_{i+1}$.
(iv) $X_i \cap X_j = \phi$ for all i and j, $i \neq j$.

The method developed above enables us to recognise a property of the optimal solution of (I) which is summarized in the following theorem:

Theorem 2:

If $\hat{X}$ is an optimal solution of (I) and $X_1 \cap S_1 = \phi$, then $\hat{X}$ is adjacent to some element of $S_3 = S_2 \backslash S_1$ [*Kirby, Love* and *Swarup*, 1972].

Proof:

According to theorem 1, $\hat{X} \in S_2$. Let $\hat{X}$ belong to the set X_N of the N^{th} best extreme points of (II). Say $\hat{X}$ is the j^{th} member of X_N i.e., it is same as X_{N_j}. Therefore, $X_i \cap S_1 = \phi$ for $i = 1,2,\ldots,N-1$. As S_3 is the set of those extreme points of $FX = f$, $X \geq 0$ which are not extreme points of $RX = t$, $X \geq 0$, it follows that $\bigcup_{i=1}^{N-1} X_i \subseteq S_3$. As X_{N_j} is adjacent to some element of $\bigcup_{i=1}^{N-1} X_i$, it follows that X_{N_j} is adjacent to an element of S_3.

This theorem shows that the method described above does not require that all the extreme points of S_2 should be examined. This is what is done in the algorithm. We examine only elements of $\bigcup_{i=1}^{N-1} X_i$ starting from X_1 (assuming that X_{N_j} is the optimal solution of (I)) untill an N^{th} best extreme solution is found which is feasible for (I). Thus points of S_3 are investigated one by one till we reach a point in S_1 where the procedure is terminated.

We have still to study the case (L. 2) where (II) has a solution but (I) has no solution. That is, $S_2 \cap S_1 = \phi$. In this case it will be necessary to examine all the extreme points of (II). As for each i, $X_i \cap S_1 = \phi$, it seems that the procedure will never stop. But this will not be the case because S_2 has a finite number of elements and further $u_i > u_{i+1}$. So after a finite number of steps, say N, it will be impossible to get $(N+1)^{\text{th}}$ best extreme point of (II).

Thus if a stage is reached when no further best extreme point solution is possible, then (I) has no solution. The fact that (II) has no further best extreme point solution at some stage (say N^{th} stage) is indicated by the fact that at that stage $H_N = \phi$.

Statement of the Algorithm:

The whole procedure is incorporated in the following algorithm.

Algorithm:

Step 1: Solve (II). If (II) has an unbounded solution, go to step 5. If (II) has a bounded solution, find X_r, B_r (starting from $r = 1$) and go to step 2.

Step 2: Determine wheter $X_r \cap S_1 \neq \phi$. If $X_r \cap S_1 \neq \phi$, terminate the process. In this case any $X \in X_r \cap S_1$ will be an optimal solution of (I) yielding value U_r. If $X_r \cap S_1 = \phi$, go to step 3.

Step 3: Find E_r and H_r (starting from $r = 1$). If $H_r = \phi$, then (I) has no solution and the procedure is terminated. If $H_r \neq \phi$, then go to step 4.

Step 4: Find U_{r+1} and determine the sets B_{r+1} and X_{r+1} and go to step 2.

Step 5: Rewrite (II) to include the constraint $1X \leq M$, where M is a sufficiently large positive number and 1 is a sum vector, and return to step 1.

Remarks:

1. However, if degeneracy of (II) is to be considered, it can be dealt with in a manner similar to the one adopted in ordinary linear fractional functional programming problem (degenerate) or linear programming problem (degenerate) [*Hadley*, w. y.].
2. Note that when an extreme point solution, say X, of (II) is reached which is not an optimal solution of (I), then we examine only those of its adjacent extreme points which yield lesser value (less than the value yielded by X) of the objective function. Such adjacent extreme points are obtained by entering those vectors in the basis of the given extreme point (i.e., X) for which Δ_j's are negative [*Swarup*, 1965]. (In numerical example, Δ_j's are defined on page 8.)

Example:

$$\text{Max } Z = \frac{2x_1 + x_2}{4x_1 + x_2 + 1}$$

$$\begin{aligned} \text{subject to } -2x_1 + x_2 &\leq 1 \\ 2x_1 + 5x_2 &\leq 23 \\ 2x_1 + x_2 &\leq 15 \end{aligned}$$

and (x_1, x_2) is an extreme point of

$$\begin{aligned} -3x_1 + 2x_2 &\leq 4 \\ x_1 + 4x_2 &\leq 22 \\ 5x_1 + 4x_2 &\leq 46 \\ x_1 - 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0. \end{aligned}$$

After adding slack variables the problem (I) of theory assumes the following form:

$$\left.\begin{array}{l} \text{Max } Z = \dfrac{2x_1 + x_2}{4x_1 + x_2 + 1} \\ \text{subject to } \left.\begin{array}{r} -2x_1 + x_2 + x_3 = 1 \\ 2x_1 + 5x_2 + x_4 = 23 \\ 2x_1 + x_2 + x_5 = 15 \end{array}\right| \equiv \Big| AX = b \\ \text{and } (x_1, x_2, \ldots, x_9) \text{ is an extreme point of} \\ \left.\begin{array}{r} -3x_1 + 2x_2 + x_6 = 4 \\ x_1 + 4x_2 + x_7 = 22 \\ 5x_1 + 4x_2 + x_8 = 46 \\ x_1 - 2x_2 + x_9 = 5 \end{array}\right| \equiv \Big| RX = t \\ x_1, x_2, \ldots, x_9 \geq 0 \end{array}\right| \quad \ldots \text{Problem (I)}$$

A is 3×9, X is 9×1, R is 4×9. Here p is $= 4$. Problem (II) of theory takes the following form

$$\left.\begin{array}{l} \text{Max } Z = \dfrac{2x_1 + x_2}{4x_1 + x_2 + 1} \\ \text{subject to } \left.\begin{array}{r} -2x_1 + x_2 + x_3 = 1 \\ 2x_1 + 5x_2 + x_4 = 23 \\ 2x_1 + x_2 + x_5 = 15 \\ -3x_1 + 2x_2 + x_6 = 4 \\ x_1 + 4x_2 + x_7 = 22 \\ 5x_1 + 4x_2 + x_8 = 46 \\ x_1 - 2x_2 + x_9 = 5 \end{array}\right| \equiv |FX = f| \\ x_1, x_2, \ldots, x_9 \geq 0 \end{array}\right| \quad \ldots \text{Problem (II)}$$

Step 1: $X_1 = [X_{11} = (3/2, 4, 0, 0, 8, \frac{1}{2}, 9/2, 45/2, 23/2)]$
$u_1 = 7/11$
$B_1 = [B_{11} = (f_2, f_1, f_5, f_6, f_7, f_8, f_9)]$

where $f_1, f_2, \ldots, f_9$ are columns of F.

$J(X_{11}) = r_1, r_2, r_6, r_7, r_8, r_9$
$|J(X_{11})| = 6 > 4$
$\therefore X_1 \cap S_1 = \phi$.

Tableau for X_{11} is:

			$c_j \rightarrow$	2	1	0	0	0	0	0	0	0
			$d_j \rightarrow$	4	1	0	0	0	0	0	0	0
$D_{B_{11}}$	$C_{B_{11}}$	Vectors in basis	$X_{B_{11}}$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
1	1	f_2	4	0	1	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0
4	2	f_1	$\frac{3}{2}$	1	0	$-\frac{5}{12}$	$\frac{1}{12}$	0	0	0	0	0
0	0	f_5	8	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	0	0	0
0	0	f_6	$\frac{1}{2}$	0	0	$-\frac{19}{12}$	$-\frac{1}{12}$	0	1	0	0	0
0	0	f_7	$\frac{9}{2}$	0	0	$-\frac{1}{4}$	$-\frac{3}{4}$	0	0	1	0	0
0	0	f_8	$\frac{45}{2}$	0	0	$\frac{17}{12}$	$-\frac{13}{12}$	0	0	0	1	0
0	0	f_9	$\frac{23}{2}$	0	0	$\frac{3}{4}$	$\frac{1}{4}$	0	0	0	0	1
$z^{(2)} = 11$,	$z^{(1)} = 7$,	$c_j - z_j^{(1)} \rightarrow$	$z = \frac{7}{11}$	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	0	0
		$d_j - z_j^{(2)}$		0	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	0	0	0
		$\Delta_j \rightarrow$		0	0	$-\frac{19}{6}$	$-\frac{1}{6}$	0	0	0	0	0

where $z^{(1)} = C_{B_{11}} X_{B_{11}}$
$z^{(2)} D_{B_{11}} X_{B_{11}} + 1$.

$C_{B_{11}}$ and $D_{B_{11}}$ are the vectors having their components as the coefficients associated with the basic variables in the numerator and denominator of the objective function.

$z_j^{(1)} = C_{B_{11}} V_j, \quad z_j^{(2)} = D_{B_{11}} V_j$
$V_j = B_{11}^{-1} f_j$

$$Z = \frac{Z^{(1)}}{Z^{(2)}}$$

$$\Delta_j = Z^{(2)} \quad (c_j - z_j^{(1)}) - z^{(1)}(d_j - z_j^{(2)}).$$

C_j is the j^{th} element of C and d_j is the j^{th} element of D.

Step 2: $E_1 = [E_{11} = (f_2,f_1,f_3,f_6,f_7,f_8,f_9), E_{12} = (f_2,f_4,f_5,f_6,f_7,f_8,f_9)]$ $\frac{15}{29}$ is the value of the objective function yielded by $(f_2,f_1,f_3,f_6,f_7,f_8,f_9)$, $\frac{1}{2}$ is the value of the objective function yield by $(f_2,f_4,f_5,f_6,f_7,f_8,f_9)$

$$\begin{aligned} u_2 &= \tfrac{15}{29} \\ B_2 &= [B_{21} = (f_2,f_1,f_3,f_6,f_7,f_8,f_9)] \\ X_2 &= [X_{21} = (\tfrac{13}{2},2,12,0,0,\tfrac{39}{2},\tfrac{15}{2},\tfrac{11}{12},\tfrac{5}{2})] \\ J(X_{21}) &= r_1,r_2,r_6,r_7,r_8,r_9 \\ |J(X_{21})| &= 6 > 4 \end{aligned}$$

$\therefore X_{21}$ is not an extreme point of $RX = t$, $X \geq 0$

$\therefore X_2 \cap S_1 = \phi$.

Step 3: $E_2 = [E_{21} = (f_2,f_1,f_3,f_6,f_7,f_8,f_4)]$

$$H_2 = E_1 \cup E_2 \backslash B_2 = [(f_2,f_4,f_5,f_6,f_7,f_8,f_9),(f_2,f_1,f_3,f_6,f_7,f_8,f_4)]$$

$\frac{1}{2}$ is the value of the objective function yielded by $(f_2,f_4,f_5,f_6,f_7,f_8,f_9)$
$\frac{1}{2}$ is the value of the objective function yielded by $(f_2,f_1,f_3,f_6,f_7,f_8,f_4)$

$$\begin{aligned} B_3 &= [B_{31} = (f_2,f_4,f_5,f_6,f_7,f_8,f_9), B_{32} = (f_2,f_1,f_3,f_6,f_7,f_8,f_4)] \\ X_3 &= [X_{31} = (0,1,0,18,14,2,18,42,7), X_{32} = (7,1,14,4,0,23,11,7,0)] \\ J(X_{31}) &= r_2,r_6,r_7,r_8,r_9 \\ |J(X_{31})| &= 5 > 4 \;\therefore X_{31} \text{ is not an extreme point of } RX = t,\ X \geq 0. \\ J(X_{32}) &= r_1,r_2,r_6,r_7,r_8 \\ |J(X_{32})| &= 5 > 4 \;\therefore X_{32} \text{ is not an extreme point of } RX = t,\ X \geq 0. \end{aligned}$$

$\therefore X_3 \cap S_1 = \phi$.

Step 4: $E_3 = [E_{31} = (f_3,f_4,f_5,f_6,f_7,f_8,f_9), E_{32} = (f_5,f_1,f_3,f_6,f_7,f_8,f_4)]$

$$\begin{aligned} H_3 &= E_1 \cup E_2 \cup E_3 \backslash B_2 \cup B_3 \\ &= [(f_3,f_4,f_5,f_6,f_7,f_8,f_9),(f_5,f_1,f_3,f_6,f_7,f_8,f_4)] \end{aligned}$$

0 is the value of the objective function yielded by $(f_3,f_4,f_5,f_6,f_7,f_8,f_9)$.
$\frac{10}{21}$ is the value of objective function yield by $(f_5,f_1,f_3,f_6,f_7,f_8,f_4)$.

$$\begin{aligned} \therefore u_4 &= \tfrac{10}{21} \\ B_4 &= [B_{41} = (f_5,f_1,f_3,f_6,f_7,f_8,f_4)] \\ X_4 &= [X_{41} = (5,0,11,13,5,19,17,21,0)] \\ J(X_{41}) &= r_1,r_6,r_7,r_8 \\ |J(X_{41})| &= 4 \end{aligned}$$

Also r_1, r_6, r_7, r_8 are linearly independent
$\therefore X_{41}$ is an extreme point $RX = t$, $X \geq 0$.

$\therefore X_4 \cap S_1 \neq \phi$.

Therefore, optimal solution of (I) is:

$$X_{41} = (5,0,11,13,5,19,17,21,0) \text{ yielding value } \tfrac{10}{21}.$$

Therefore, optimal solution of the original problem is

$$x_1 = 5, \quad x_2 = 0$$

and the optimal value is $\frac{10}{21}$.

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